

An explicit time variable for cosmology and the matter-vacuum energy coincidence.

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Abstract

By allowing for non zero vacuum expectation values for some of the fields that appear in the Hamiltonian constraint of canonical general relativity a time variable, with usual properties, can be identified; the constraint plays the role of the ordinary Hamiltonian. The energy eigenvalues contribute to the variation of the scale parameter similarly to the way matter density does. For a universe described by a superposition of eigenstates or by a thermodynamic ensemble the dominant contribution comes from energy, or equivalently effective matter density, of the same order as the vacuum energy (cosmological constant). This may explain the observed “coincidence” of these two values.

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Canonical Hamiltonian gravity, both classical and quantum, is a theory of constraints. A normal unitary evolution can be obtained only after some dynamical variable, whose canonical conjugate will play the role of time, is identified and solved for as a function of the other dynamical variables. Although this is the path to be followed in this work, it should be mentioned that it is not the universal approach to the problem of time in cosmology. In the ‘no boundary’ approach of Hartle and Hawking [1] and in the ‘nucleation from a point’ view of Vilenkin [2] the existence or nonexistence of time is ignored and only correlations between dynamical variables are looked for. In the extrinsic time approach, time is related to a particular foliation of a three geometry in four dimensional space-time; some recent works [3] have used this method. Extensive reviews may be found in [4, 5] and more recently in [6]. As mentioned, in this work we shall look for a dynamical variable that can be identified as time. Some of the difficulties that this approach has had are overcome in that we allow for some fields to acquire non-zero expectation values before the constraint equations are solved. This will result in a Schrödinger equation with the Hamiltonian constraint playing the role of the Hamiltonian itself. Applying this to the evolution of the scale $R(t)$ in cosmological models, we find that the energy eigenvalues E contribute in the same functional form as the matter density, which varies as $1/R^3$ but is multiplied by R^3 . A universe, whose wave function is a superposition of such eigenstates, may result in the effective matter density and the vacuum energy (cosmological constant) tracking each other. This would explain the present “coincidence” of these two values [7]. In obtaining these results we use the minisuperspace approximation; at each step, however, we show how the terms we introduce arise from a Lagrangian fully compatible with general relativity.

Hamiltonian gravity and cosmology are usually treated in the, afore mentioned, minisuperspace approximation, where the continuous set of gravitational and other degrees of freedom are reduced to a small number of collective, spatially constant fields. In the simplest case, the dynamics of a Robertson-Walker-Friedman universe, with a metric

$$ds^2 = N^2(t)dt^2 - R^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right], \quad (1)$$

are specified by the action for the *one* variable $R(t)$

$$S = \int [P_R dR - N dt \mathcal{H}_G(P_R, R)] . \quad (2)$$

with P_R being the momentum conjugate to R and

$$\mathcal{H}_G = -\frac{G}{3\pi R}P_R^2 - \frac{3\pi}{4G}\left(kR - \frac{\Lambda R^3}{3}\right) + 2\pi^2\rho(R)R^3; \quad (3)$$

G is Newton's constant, Λ is the cosmological constant and $\rho(R)$ accounts for the matter and radiation density. Varying this action with respect to the lapse function N yields the constraint $\mathcal{H}_G(P_R, R) = 0$ and setting the variation of $\mathcal{H}_G(P_R, R)$ with respect to P_R and R to zero insures the conservation of the energy-momentum stress tensor; there is no room for both time and a dynamical variable that depends on it.

To allow for the introduction of time more fields have to appear. A Lagrangian for a scalar field τ ,

$$\mathcal{L}_\tau = \int d^4x \sqrt{-g} \frac{g^{\mu\nu}}{2} \partial_\mu \tau \partial_\nu \tau, \quad (4)$$

can be added to the one for gravity; in the minisuperspace approximation this changes eq. (2) to

$$S = \int \left[P_R dR + \pi_\tau d\tau - N dt \left(\mathcal{H}_G + \frac{\pi_\tau^2}{2R^3} \right) \right], \quad (5)$$

where π_τ is the momentum conjugate to τ . One then tries to make τ play the role of time for R or vice versa. A serious drawback of this approach is that the resulting Schrödinger equation is hyperbolic and problems similar to those that occur in the first quantized treatment of the Klein-Gordon equation also occur here. These problems would go away if π_τ were to appear *linearly* in the coefficient of N in eq. (5). This is a goal of this work.

To achieve this end we add to the following to the Lagrangian for pure gravity:

$$\begin{aligned} \mathcal{L}_t = \int d^4x \left\{ \sqrt{-g} \left[\frac{g^{\mu\nu}}{2} \partial_\mu \tau \partial_\nu \tau + b g^{\mu\nu} \partial_\mu \tau \partial_\nu \chi - c (g^{\nu\nu'} g^{\lambda\lambda'} g^{\sigma\sigma'} H_{\nu\lambda\sigma} H_{\nu'\lambda'\sigma'} + h^2)^2 \right] \right. \\ \left. + d \epsilon^{\mu\nu\lambda\sigma} \partial_\mu \chi H_{\nu\lambda\sigma} \right\}. \end{aligned} \quad (6)$$

τ is the field whose spatially constant part will ultimately play the role of time. χ is a cyclic field, $0 \leq \chi < 2\pi$ and $H_{\nu\lambda\sigma}$ is an antisymmetric three indexed tensor field. With h^2 positive the third term in eq. (6) ensures that the spatial parts of H acquire an expectation value. It should be noted that due to the $\epsilon^{\mu\nu\lambda\sigma}$ in the coefficient of d , no $\sqrt{-g}$ is necessary for this term and, by design, no $(\partial\chi)^2$ appears in the Lagrangian; the cyclic nature of the field χ can result from χ being the phase of a regular complex field whose magnitude is fixed to some vacuum expectation value. A kinetic energy for $H_{\nu\lambda\sigma}$ can occur as can a potential involving τ . We will show that this produces a π_τ appearing linearly in the constraint.

The minisuperspace approximation to (6) is

$$\mathcal{L}_{tMS} = \int dt \left[\frac{R^3}{2N} \left(\frac{d\tau}{dt} \right)^2 + b \frac{R^3}{N} \frac{d\tau}{dt} \frac{d\chi}{dt} - cNR^3 \left(H_{ijk}H_{ijk}/R^6 - h^2 \right)^2 + d \frac{d\chi}{dt} \epsilon^{ijk} H_{ijk} \right]. \quad (7)$$

Replacing H_{ijk} by its vacuum expectation value

$$\langle H_{ijk} \rangle = \frac{R^3 h \epsilon_{ijk}}{6}. \quad (8)$$

the Hamiltonian corresponding to \mathcal{L}_{tMS} is

$$\mathcal{H}_{tMS} = \frac{N}{b^2 R^3} \left[b\pi_\tau (\pi_\chi - dhR^3) - \frac{1}{2} (\pi_\chi - dhR^3)^2 \right]. \quad (9)$$

In the fixed $\pi_\chi = 0$ (a small value for the parameter b would favor $\pi_\chi = 0$) sector we recover a term linear in π_τ . Rescaling τ we obtain, instead of the Wheeler-de-Witt equation, a Schrödinger equation with \mathcal{H}_G acting as a normal Hamiltonian [8],

$$-i \frac{\partial}{\partial \tau} + \mathcal{H}_G = 0. \quad (10)$$

The contribution of an eigenvalue E of eq. (10) for \mathcal{H}_G corresponding to eq. (3) cannot be distinguished from the contributions of matter, $\rho_M(R) \sim 1/R^3$, to the evolution of the scale parameter R ; thus we may consider $(E - 2\pi^2 \rho_M R^3)$ to be the effective energy eigenvalue, or equivalently $(\rho_M - E/2\pi R^3)$ the effective matter density. Interesting results obtain for a universe that, rather than being a state with a definite energy, is a superposition of such eigenstates

$$\Psi(R, \tau) = \int dE f(E) \psi_E(R) e^{-iE\tau}; \quad (11)$$

$\psi_E(R)$ is an eigenfunction of \mathcal{H}_G and $f(E)$ describes the energy wave packet. The possibility now arises that at different R 's different E 's dominate the integral in eq. (11). Should the E and R dependence $\ln \psi_E(R)$ appear in the combination

$$\ln \psi_E(R) = R^3 g \left(\frac{\Lambda R^3}{E} \right), \quad (12)$$

with g some function and with $\ln[f(E)]$ varying no faster than $c|E|$, the dominant contribution to $\Psi(R, \tau)$ in eq. (11) comes from $E \sim \Lambda R^3$. As was remarked earlier, E contributes to the evolution of R the same way as $\rho_M R^3$ and thus *for any given R the effective $\rho_M \sim \Lambda$* . Thus the observation that today $\rho_M \sim \Lambda$ [7] may not be a coincidence but may hold true at all scale factors R [9].

An example of $\psi_E(R)$ scaling according to eq. (12) is the WKB approximation for the observationally favored case of a flat, $k = 0$, universe [7] in an epoch where the radiation density may be neglected; ψ_E is a combination of

$$\psi_E^{(\pm)}(R) = \exp \left(\pm \frac{3\pi}{2G} i \int_0^R dr \sqrt{\frac{\Lambda r^4}{3} - \frac{4GEr}{3\pi}} \right). \quad (13)$$

The integrals in the exponents of the above have the desired scaling property. For a universe described by a thermodynamic density matrix at a temperature $1/\beta$ the dominant contributions to the diagonal density matrix elements, those that account for the distribution of the scale factor R ,

$$\rho(R, R) = \int dE \psi_E(R) e^{-\beta E} \psi_E^*(R), \quad (14)$$

likewise come from $E \sim \Lambda R^3$.

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 - [8] The last term in eq. (9) contributes to the cosmological constant.
 - [9] An alternate suggestion that it is the effective cosmological constant that varies has recently been made by K. Griest [astro-ph/0202052].